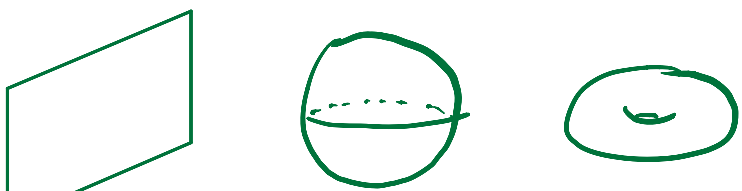
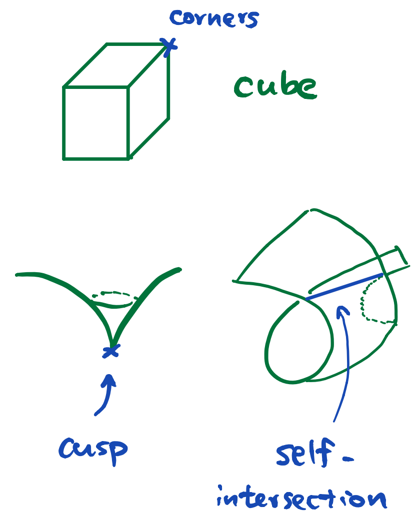
§ Surfaces in $\mathbb{R}^3$

Q: Which one do we want to consider as "surfaces"?



plane sphere torus

YES



NO

Basic idea: A "surface" is an object that ① locally ② looks like a piece of $\mathbb{R}^2$.

Definition: A (regular) surface is a subset

$$S \subseteq \mathbb{R}^3$$

s.t. $\forall p \in S$, $\exists$ nbd. of p (in S) $V \subseteq S$

and a smooth map (called parametrization / chart)

$$\mathbf{X} : \underset{\text{open}}{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow V$$

s.t. (1) $\mathbf{X} : \mathcal{U} \to V$ is a homeomorphism.

* (2) The differential $d\mathbf{X}|_q$ is 1-1 $\forall q \in \mathcal{U}$.

Explanation on $*$: More explicitly,

$$\mathbf{X}(u,v) = \big(x(u,v), y(u,v), z(u,v)\big), \quad q = (u,v) \in \mathcal{U}$$

smooth functions

The differential of $\mathbf{X}$ at $q \in \mathcal{U}$ is a linear map

$$d\mathbf{X}\big|_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

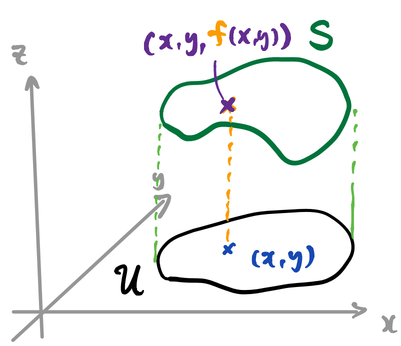
which can be expressed in matrix form (w.r.t. std basis)

$$d\mathbf{X}\big|_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}\Bigg|_q =: \begin{pmatrix} | & | \\ \frac{\partial \mathbf{X}}{\partial u} & \frac{\partial \mathbf{X}}{\partial v} \\ | & | \end{pmatrix}\Bigg|_q$$

$$\boxed{d\mathbf{X}\big|_q \text{ is 1-1} \iff \frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v} \text{ are linearly independent.}}$$

Example 1 : Graphical surfaces

Given a smooth function $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$



$$S = \text{graph}(f) = \{ z = f(x,y) \mid (x,y) \in \mathcal{U} \}$$

is a (regular) surface.

Why? Consider the smooth map

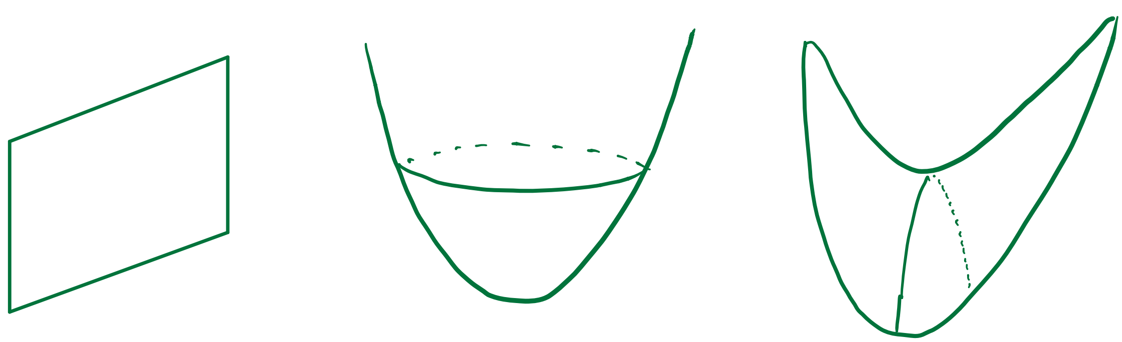
$$\begin{array}{ccc} \mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 & \longrightarrow & S \subseteq \mathbb{R}^3 \\ \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} \\ (u,v) & \longmapsto & (u, v, f(u,v)) \end{array}$$

Clearly, $\mathbf{X} : \mathcal{U} \to S$ is a homeomorphism.

$$\left.\begin{aligned} \frac{\partial \mathbf{X}}{\partial u} &= \left(1, 0, \frac{\partial f}{\partial u}\right) \\ \frac{\partial \mathbf{X}}{\partial v} &= \left(0, 1, \frac{\partial f}{\partial v}\right) \end{aligned}\right\} \text{ always linearly independent.}$$

Examples of graphical surfaces

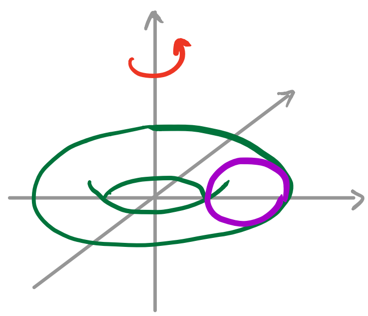
- $f(x,y) = ax + by + c \implies$ planes
- $f(x,y) = x^2 + y^2 \implies$ paraboloid
- $f(x,y) = x^2 - y^2 \implies$ hyperboloid



plane paraboloid hyperboloid

Note: The entire surface can be covered by 1 chart.

Example 2 : Torus of revolution



$$S = \{ (\sqrt{x^2+y^2} - a)^2 + z^2 = r^2 \}$$

where $a > r > 0$ are constants.

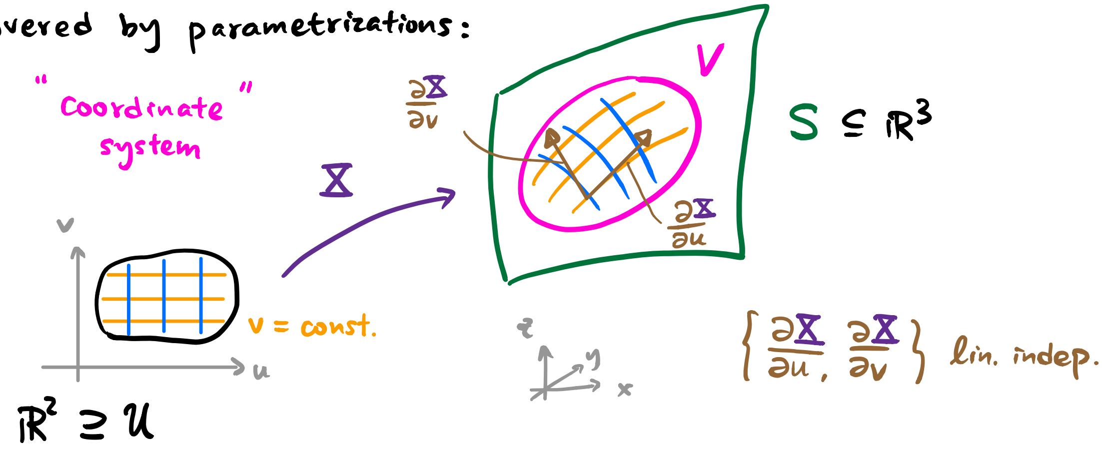
is a surface.

Exercise: Prove this.

How many parametrizations do we need to cover the whole torus?

§ Differential Calculus on surfaces

Recall that a **surface** is a subset $S \subseteq \mathbb{R}^3$ covered by parametrizations:



Examples: Spheres, torus, graphs $z = f(x,y)$ ......

Goal: Do calculus on surfaces.

Let's recall two important theorems from multivariable calculus.

Inverse Function Theorem

Let $F : \underset{\text{open}}{\mathcal{U} \subseteq \mathbb{R}^n} \longrightarrow \mathbb{R}^n$ be a smooth map. $p_0 \in \mathcal{U}$.

Suppose $dF|_{p_0} : \mathbb{R}^n \to \mathbb{R}^n$ is a linear isomorphism.

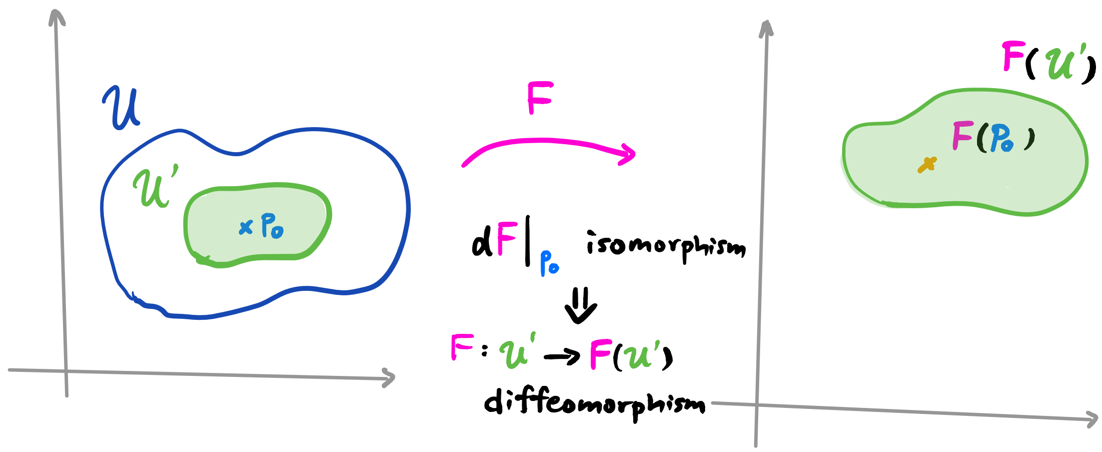
Then, F is a local diffeomorphism near p_0.

i.e. $\exists$ nbd $\mathcal{U}' \subseteq \mathcal{U}$ of p_0 s.t.

$$F|_{\mathcal{U}'} : \mathcal{U}' \longrightarrow F(\mathcal{U}')$$

is a diffeomorphism, i.e. smooth, bijective with smooth inverse.

Inverse function theorem



Implicit Function Theorem

Let $F = F(x,y,z) : \mathcal{O} \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function.

Consider the level surface of F at $a \in \mathbb{R}$

$$F^{-1}(a) := \{ p \in \mathcal{O} : F(p) = a \}.$$

Suppose $p_0 = (x_0, y_0, z_0) \in F^{-1}(a)$ and $\boxed{\left.\frac{\partial F}{\partial z}\right|_{p_0} \neq 0}$.

Then, $\exists$ a nbd $V \subseteq \mathbb{R}^3$ of p_0 & a smooth function

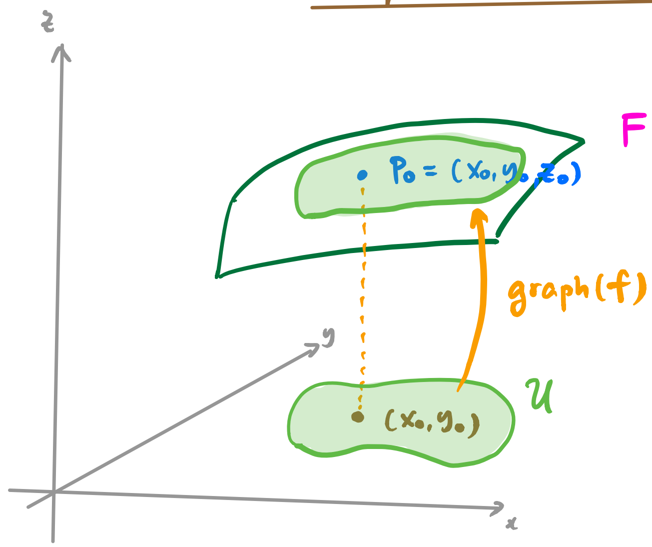
$$f : \underset{\text{open}}{\mathcal{U} \subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}$$

$$(x_0, y_0) \longmapsto z_0$$

s.t. $$F^{-1}(a) \cap V = \operatorname{graph}(f)$$

$$= \{ (x, y, f(x,y)) : (x,y) \in \mathcal{U} \}.$$

Implicit function theorem



$$\boxed{\left.\frac{\partial F}{\partial z}\right|_{P_0} \neq 0}$$

Proposition: Any surface $S \subseteq \mathbb{R}^3$ is locally a graph,

i.e. for each $p \in S$, $\exists$ nbd V of p in S s.t.

$$V = \{z = f(x,y)\} \text{ or } \{y = f(x,z)\} \text{ or } \{x = f(y,z)\}$$

for some smooth function $f : \mathcal{U} \subseteq \mathbb{R}^2 \to \mathbb{R}$.

Proof: Fix $p \in S$, $\exists$ parametrization

$$\begin{array}{ccccc} \mathbf{X} : & \mathcal{U} \subseteq \mathbb{R}^2 & \xrightarrow{\cong} & V \subset S, & \mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v)) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & \\ & q & \longmapsto & p & \end{array}$$

s.t.

$$d\mathbf{X}\big|_q = \left.\begin{pmatrix} \boxed{\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}} \\ \begin{matrix} \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{matrix} \end{pmatrix}\right|_q : \mathbb{R}^2 \to \mathbb{R}^3$$

is 1-1

($\Leftrightarrow$ rank = 2)

($\Leftrightarrow$ $\exists$ 2×2 invertible submatrix)

By Inverse Function Theorem, locally we can solve u, v in terms of x, y, i.e.

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases} \Rightarrow$$ locally near p in S is the graph of the function $z(u(x,y), v(x,y))$

$\square$

Given a smooth function $F: \mathbb{R}^3 \to \mathbb{R}$, for each $a \in \mathbb{R}$, consider the level set

$$F^{-1}(a) := \{ p \in \mathbb{R}^3 : F(p) = a \}$$

Question: When is it a surface?

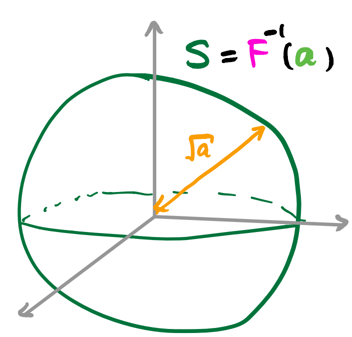
Def$^{\underline{n}}$: a is a regular value of F if $\forall p \in F^{-1}(a)$, $\nabla F|_p \neq 0$.

Theorem: $F^{-1}(a)$ is a surface for any regular value a of F.

Proof: The implicit function theorem implies that $F^{-1}(a)$ is locally a graph, hence it is a surface. $\square$

Example: $F(x,y,z) = x^2 + y^2 + z^2$

$F^{-1}(a)$ = Sphere of radius $\sqrt{a}$

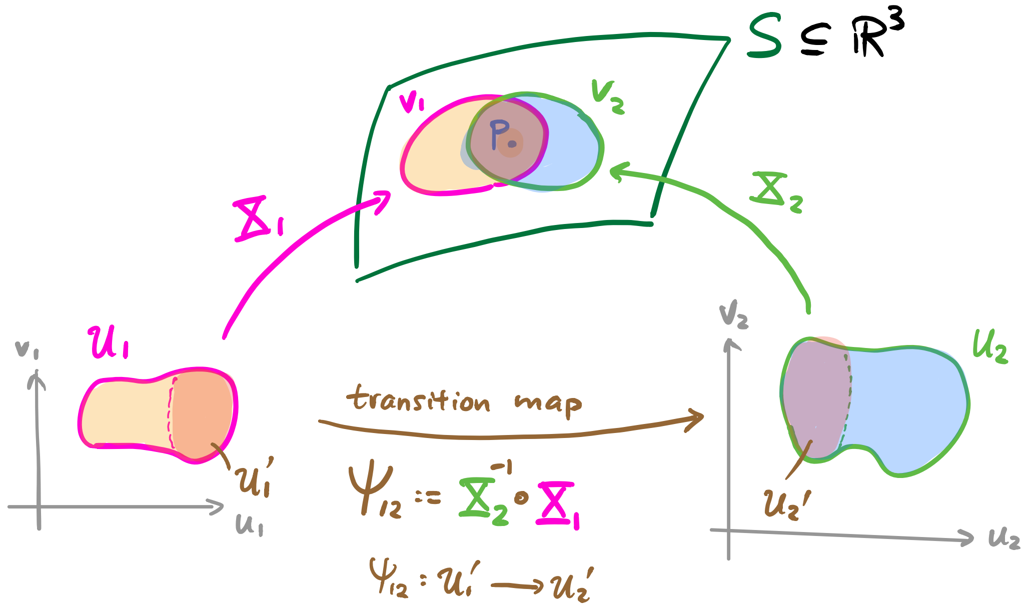


"Singular" when $a = 0$ since

$$\nabla F(0) = (2x, 2y, 2z)\Big|_0 = 0.$$

§ Change of parameters / coordinates

Consider a point $p \in S$ covered by two coordinate systems



Claim: $\varphi_{12} : U_1' \to U_2'$ is a diffeomorphism (between open sets in $\mathbb{R}^2$)

Proof: Clearly, φ_{12} is a homeomorphism with

inverse $\varphi_{12}^{-1} = \varphi_{21} := \mathbf{X}_1^{-1} \circ \mathbf{X}_2 : U_2' \to U_1'$

It remains to show that φ_{12} is smooth.

Since S is locally a graph, say over xy-plane.

Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection map onto xy-plane.

Then,
$$\varphi_{12} = \mathbf{X}_2^{-1} \circ \mathbf{X}_1 = \underbrace{(\pi \circ \mathbf{X}_2)^{-1}}_{\mathbb{R}^2 \to \mathbb{R}^2} \circ \underbrace{\pi}_{\mathbb{R}^3 \to \mathbb{R}^2} \circ \underbrace{\mathbf{X}_1}_{\mathbb{R}^2 \to \mathbb{R}^3} \quad \text{Smooth!}$$

smooth maps on open subsets of

$\blacksquare$

§ Differentiability

We now define the concept of "differentiability" of functions into and out of a surface $S \subseteq \mathbb{R}^3$.

<u>Def$^{\underline{n}}$</u>: A function $f: S \longrightarrow \mathbb{R}^n$ is **smooth**

if $f \circ \mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}^n$ is smooth

for <u>ANY</u> parametrization $\mathbf{X}: \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$

<u>Remark</u>: Since change of coordinates Ψ_{12} are smooth diffeomorphisms, we just have to check smoothness on <u>SOME</u> collection of parametrizations covering S.

<u>Example</u>: (Restriction of smooth functions on $\mathbb{R}^3$)

Let $f: V \overset{\text{open}}{\subseteq} \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function

s.t. $S \subseteq V$, then

$$f\big|_S : S \longrightarrow \mathbb{R} \quad \text{is smooth.}$$

In particular, the restrict of the coordinate functions x, y and z to S are smooth functions.

<u>FACT</u>: $\underset{\text{smooth}}{f, g : S \to \mathbb{R}} \implies \underset{\text{smooth}}{f \pm g,\ fg,\ f/g}$ (if well-defined)

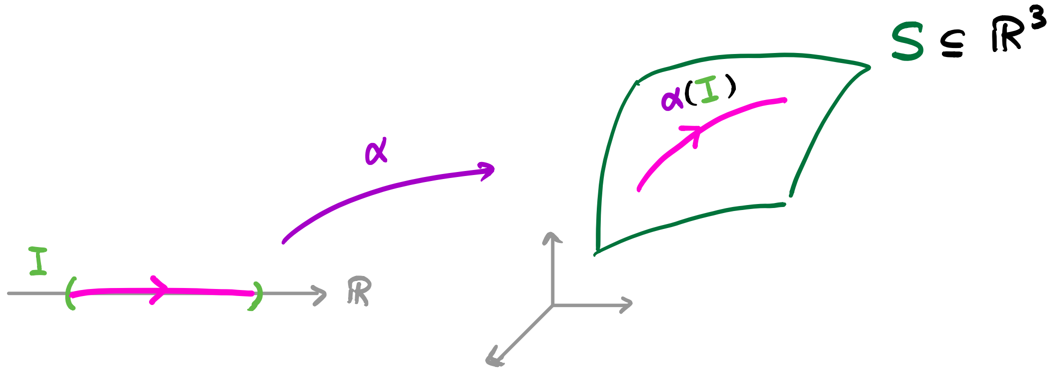
Def$^{\underline{n}}$: A map $f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow S$ is smooth

if it is smooth as a map $f : \mathcal{U} \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^3$.

Example: Curves on a surface

Let $\alpha : I \longrightarrow \mathbb{R}^3$ be a space curve with $\alpha(I) \subseteq S$

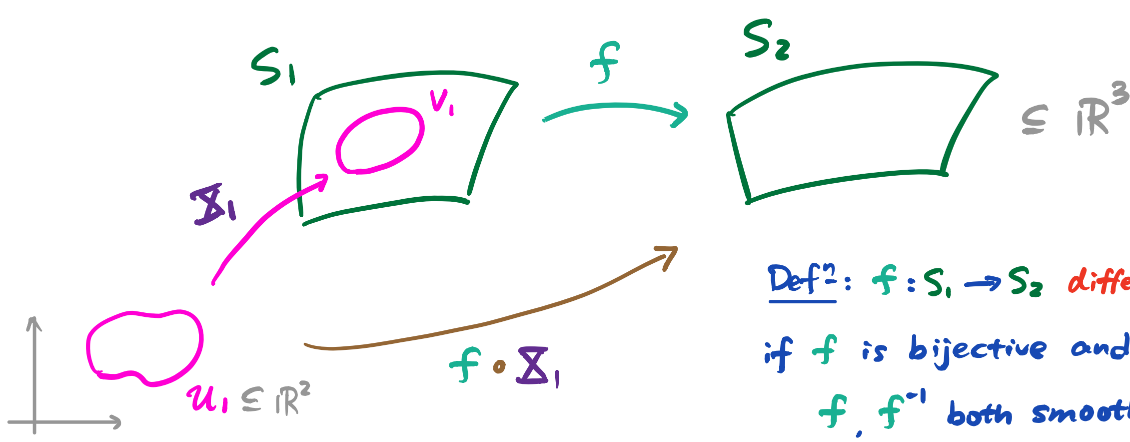
Then, $\alpha : I \subseteq \mathbb{R} \longrightarrow S$ is smooth.



Def$^{\underline{n}}$: A map $f : S_1 \longrightarrow S_2$ between two surfaces is

smooth if $f \circ \mathbf{X}_1 : \mathcal{U}_1 \subseteq \mathbb{R}^2 \longrightarrow S_2 \subseteq \mathbb{R}^3$ is smooth

for any parametrization $\mathbf{X}_1$ of S_1.



Def$^{\underline{n}}$: $f : S_1 \rightarrow S_2$ diffeomorphism

if f is bijective and

f, f^{-1} both smooth.

§ Tangent planes & the differential of a map

Recall: For a smooth map $f : \mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{R}^n \to \mathbb{R}^m$, $p \in \mathcal{U}$,

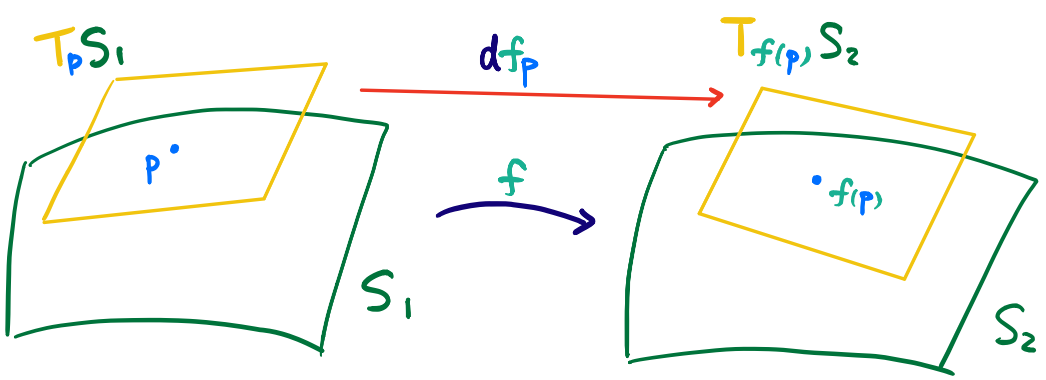
differential $\quad df_p : \mathbb{R}^n \longrightarrow \mathbb{R}^m \quad$ linear map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \\ \cong & & \cong \\ T_p\mathcal{U} & & T_{f(p)}\mathbb{R}^m \end{array}$$

If $f : S_1 \longrightarrow S_2$ is a smooth map between surfaces,

differential of f at $p \in S_1$

$$\boxed{df_p : T_pS_1 \longrightarrow T_{f(p)}S_2} \quad \text{linear}$$



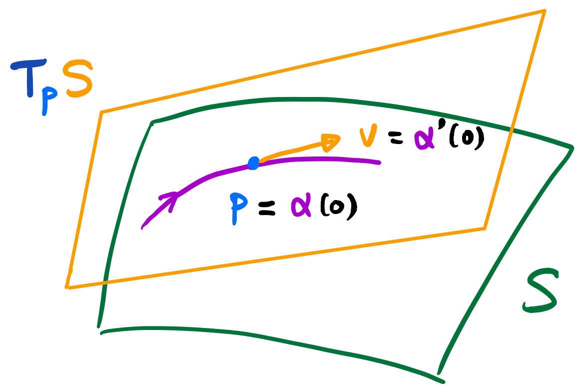
Question: How to define it ?

There are many (but equivalent) ways to define it.

We will use curves on surfaces.

Def$^{\underline{n}}$: The tangent plane of S at $p \in S$

$$T_pS := \left\{ v \in \mathbb{R}^3 : \begin{array}{l} \overset{\text{not unique!}}{\exists} \text{ smooth curve } \alpha : (-\varepsilon, \varepsilon) \to S \\ \text{s.t. } \alpha(0) = p \text{ and } \alpha'(0) = v \end{array} \right\}$$

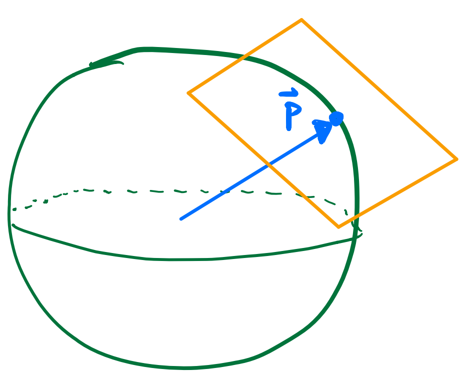


Example: $\underset{p \,\in}{S} = F^{-1}(a)$ for some regular value $a \in \mathbb{R}$

of a smooth $F : \underset{\text{open}}{O \subseteq} \mathbb{R}^3 \to \mathbb{R}$.

Then, $\quad T_pS = (\underbrace{\nabla F|_p}_{\neq \vec{0}})^{\perp}$ by "advanced calculus"

E.g. Sphere $\quad S = \mathbb{S}^2(r) := \{ x^2 + y^2 + z^2 = r^2 \}$.



$T_pS \perp \vec{p}$ $\quad \leftarrow$ position vector at p

Exercise: Prove this!

Def$^{\underline{n}}$: Let $f: S_1 \to S_2$ be a smooth map between surfaces.

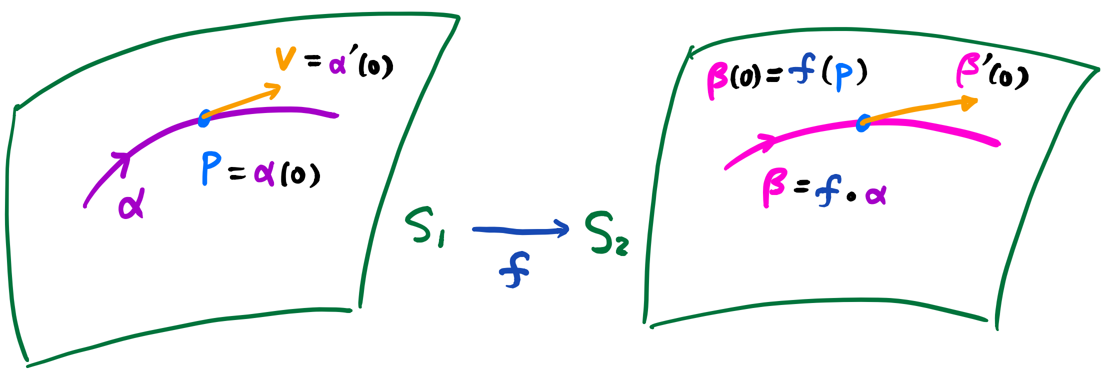
The differential of f at $p \in S_1$ is a map

$$df_p : T_pS_1 \longrightarrow T_{f(p)}S_2$$

defined as follows: for each $v \in T_pS_1$, choose any curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S_1$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

let $\beta = f \circ \alpha : (-\varepsilon, \varepsilon) \longrightarrow S_2$ with $\beta(0) = f(p)$

$$df_p(v) := \beta'(0) \in T_{f(p)}S_2$$



Remark: We need to check that df_p is well-defined (i.e. independent of the choice of α) and that it is a linear map between the tangent planes.

§ Tangent Planes & Differential

Recall that T_pS and $df_p : T_pS_1 \longrightarrow T_{f(p)}S_2$ are defined using curves on surfaces:

$$T_pS := \left\{ v \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} \exists\ \alpha : (-\varepsilon, \varepsilon) \to S \text{ s.t.} \\ \alpha(0) = p\ ,\ \alpha'(0) = v \end{array} \right\}$$

and $df_p(\alpha'(0)) := (f \circ \alpha)'(0)$ well-defined? linear?

<u>Lemma:</u> If $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$ is <u>ANY</u> parametrization s.t. $\mathbf{X}(q) = p \in S$, then

$$\boxed{T_pS = d\mathbf{X}_q(\mathbb{R}^2) = \operatorname{span}\left\{ \frac{\partial \mathbf{X}}{\partial u}, \frac{\partial \mathbf{X}}{\partial v} \right\}}$$

<u>Proof:</u> Take any $v \in T_pS$, by definition

$\exists\ \alpha : (-\varepsilon, \varepsilon) \longrightarrow S$ s.t. $\alpha(0) = p$, $\alpha'(0) = v$

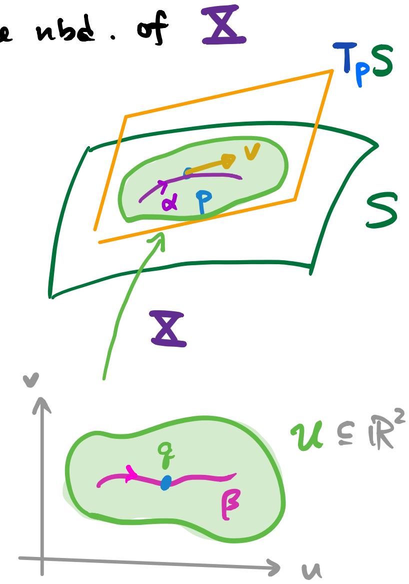
Restricting $\varepsilon > 0$ small if necessary, we can assume that α lies completely inside the coordinate nbd. of $\mathbf{X}$

$\Rightarrow \exists\ \beta : (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U} \subseteq \mathbb{R}^2$ s.t.

$$\alpha = \mathbf{X} \circ \beta$$

By Chain Rule,

$$v = \alpha'(0) = \underbrace{d\mathbf{X}_q(\beta'(0))}_{\in\, d\mathbf{X}_q(\mathbb{R}^2)}$$



We will often write a tangent vector as

$$\boxed{v = a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v}}$$

as directional derivative $\quad v(f) := a\frac{\partial f}{\partial u} + b\frac{\partial f}{\partial v}$ where $f \in C^\infty(S)$

(In fact, we can define tangent vectors as operators by directional derivatives.)

<u>Lemma</u>: df_p is well-defined & linear.

<u>Proof</u>: Take any parametrization of S_1 near p:

$$\begin{array}{cccc} X: & \mathcal{U} \subseteq \mathbb{R}^2 & \longrightarrow & S_1 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & q & & p = X(q) \end{array}$$

By previous lemma,

$$dX_q : \mathbb{R}^2 \xrightarrow[\text{linear isomorphism}]{\cong} T_pS_1 \subseteq \mathbb{R}^3$$

$\rightsquigarrow \quad (dX_q)^{-1} : T_pS_1 \longrightarrow \mathbb{R}^2 \quad$ exists.

Then one can check easily:

$$df_p = \underbrace{\underset{\mathbb{R}^2 \to T_{f(p)}S_2 \subseteq \mathbb{R}^3}{d(f \circ X)_q} \circ \underset{T_pS_1 \to \mathbb{R}^2}{(dX_q)^{-1}}}_{\text{well-defined \& linear}}$$

$\square$

Properties of differential:

(1) Chain Rule : $\boxed{d(g \circ f)_p = dg_{f(p)} \cdot df_p}$

(2) Inverse Function Theorem

Any smooth $f: S_1 \to S_2$ with

$$df_p : T_p S_1 \xrightarrow{\cong} T_{f(p)} S_2 \quad \text{linear isomorphism}$$

is a local diffeomorphism near p.

(3) If $f: S \to \mathbb{R}^m$ has $df_p = 0$ for all $p \in S$ and S is connected, then $f \equiv$ constant.

(3) If $f: S \to \mathbb{R}$ has a local max/min at $p \in S$ then $df_p = 0$.

Proof: Exercises!

In other words, all the familiar facts in calculus hold for functions defined on surfaces.

We now look at an interesting application of this concept in Linear Algebra.

Application (Spectral Theorem)

Show that any symmetric $A \in M_{3\times3}(\mathbb{R})$ has an orthonormal eigenbasis.

Proof: Consider the function on the unit sphere $S^2 \subseteq \mathbb{R}^3$:

$$\begin{aligned} f : S^2 &\longrightarrow \mathbb{R} \\ p &\longmapsto \langle Ap, p\rangle \end{aligned}$$

smooth!

Lemma: $df_p = 0 \iff Ap = f(p)\,p$

i.e. $p \in S^2$ is eigenvector of A with eigenvalue $f(p)$

Proof: Note that f is the restriction to S^2 of a smooth function:

$$\begin{aligned} F : \mathbb{R}^3 &\longrightarrow \mathbb{R} \\ p &\longmapsto \langle Ap, p\rangle \end{aligned}$$

If we write $p = (x, y, z)$, then F is a homogeneous quadratic polynomial in x, y, z, hence

$$dF_p(v) = 2\langle Ap, v\rangle \qquad \forall\, v \in \mathbb{R}^3$$

restrict to T_pS^2 $\Rightarrow$

$$df_p : T_pS^2 \longrightarrow \mathbb{R}, \quad df_p(v) = 2\langle Ap, v\rangle$$

$$T_pS^2 = \{v \in \mathbb{R}^3 : v \perp p\}$$

Therefore, $df_p = 0$

$\iff \langle Ap, v \rangle = 0$ for all $v \in \mathbb{R}^3$ s.t. $v \perp p$

$\iff \quad Ap = \lambda p$ for some $\lambda \in \mathbb{R}$

$$\left(f(p) = \langle Ap, p \rangle = \langle \lambda p, p \rangle = \lambda \right)$$

This proves the lemma.

If $f \equiv$ constant, then $A = \lambda I$ (trivial)

If $f \not\equiv$ constant, then since S^2 is compact

$\exists$ $p_1 \neq p_2 \in S^2$ s.t.

$$f(p_1) = \max_{S^2} f > f(p_2) = \min_{S^2} f$$

A symmetric $\Rightarrow$ $p_1 \perp p_2$ $\rightsquigarrow$ $p_3 := p_1 \times p_2$

Claim: p_3 is an eigenvector of A

Pf: $\langle Ap_3, p_1 \rangle = \langle p_3, Ap_1 \rangle = f(p_1) \underbrace{\langle p_3, p_1 \rangle}_{=0} = 0$

Similarly, $\langle Ap_3, p_2 \rangle = 0$.

Therefore, $Ap_3 \parallel p_3$.

Hence, $\{p_1, p_2, p_3\}$ is an orthonormal eigenbasis

for the symmetric matrix A.

$\blacksquare$

§ Vector fields on surfaces

Def$^{\underline{n}}$: A vector field on S is just a smooth map

$$V : S \longrightarrow \mathbb{R}^3$$

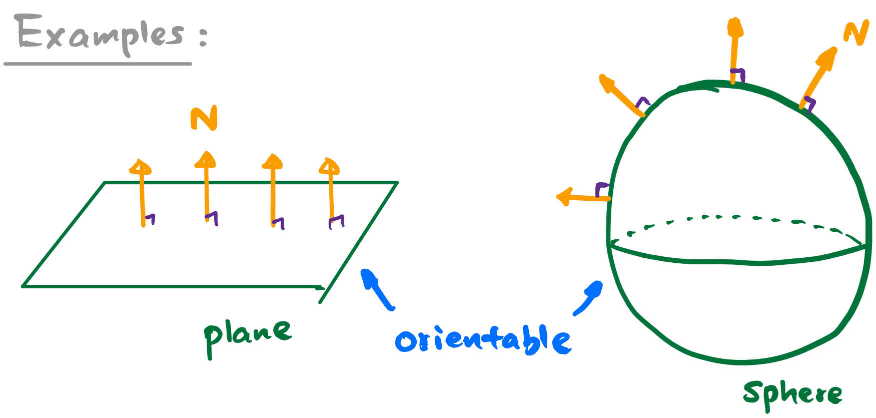
- V tangential if $V(p) \in T_pS \quad \forall p \in S$
- V normal if $V(p) \perp T_pS \quad \forall p \in S$

Def$^{\underline{n}}$: A surface $S \subseteq \mathbb{R}^3$ is orientable

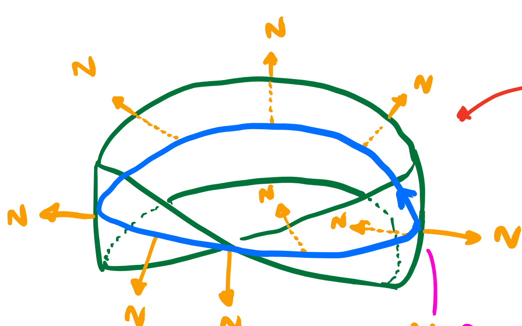
if $\exists$ unit normal vector field $N : S \to \mathbb{R}^3$

i.e. $N(p) \perp T_pS$, $\|N(p)\| = 1 \quad \forall p \in S$

Examples:



"Mobius strip"



NOT orientable!

N flips after travelling around the central circle once.

Remarks: (1) Any orientable surface S has exactly 2 distinct orientations, N and $-N$

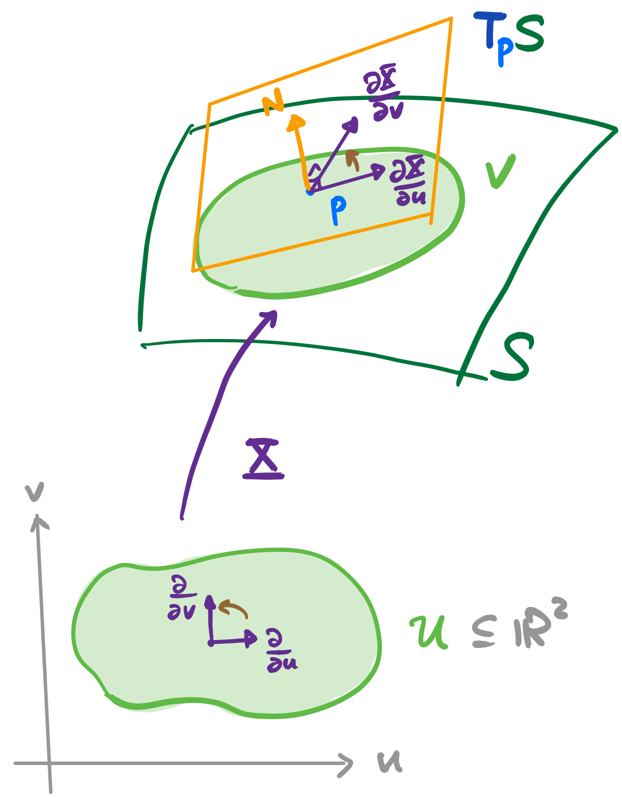
(2) Orientability is a global property.

FACT: Any surface is "locally orientable".

Fix a chart $X : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow V \subseteq S$

$$\boxed{N := \frac{\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}}{\left\| \frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \right\|}}$$

unit normal vector field defined locally on V



Ex: Can you find another parametrization that induces the opposite orientation?

(3) One can also define orientability from an intrinsic point of view:

A smooth n-manifold M is orientable if $\exists$ collection of charts $\{ X_\alpha : U_\alpha \subseteq \mathbb{R}^n \to M \}$ s.t.

$\bigcup_\alpha X_\alpha(U_\alpha) = M$ and the transition maps

$$\psi_{\alpha\beta} = X_\beta^{-1} \circ X_\alpha : U_\alpha \xrightarrow{\cong} U_\beta$$

are orientation preserving diffeomorphisms.

Examples of orientable surfaces

(1) Graphical surfaces

$$S = \text{graph}(f) \qquad \text{for } f : U \subseteq \mathbb{R}^2 \to \mathbb{R} \text{ smooth}$$

(2) Level surfaces

$$S = F^{-1}(a)$$

where $a \in \mathbb{R}$ is a regular value of $F : \mathbb{R}^3 \to \mathbb{R}$ smooth

Exercise: Prove that they are orientable surfaces.

§ Differential topology of surfaces

<u>Defⁿ</u>: A surface $S \subseteq \mathbb{R}^3$ is closed if S is compact without boundary.

Classification of surfaces

A closed orientable surface is homeomorphic to one and only one of the following:

(*)

genus (i.e. number of holes) :	0	1	2	…	g
Euler characteristic χ :	2	0	-2	…	$2-2g$

Remark: There is also a classification of non-orientable closed surfaces given by the real projective plane, Klein bottle, etc……

We state two important theorems whose proof will be omited here.

Jordan Brouwer Separation Theorem

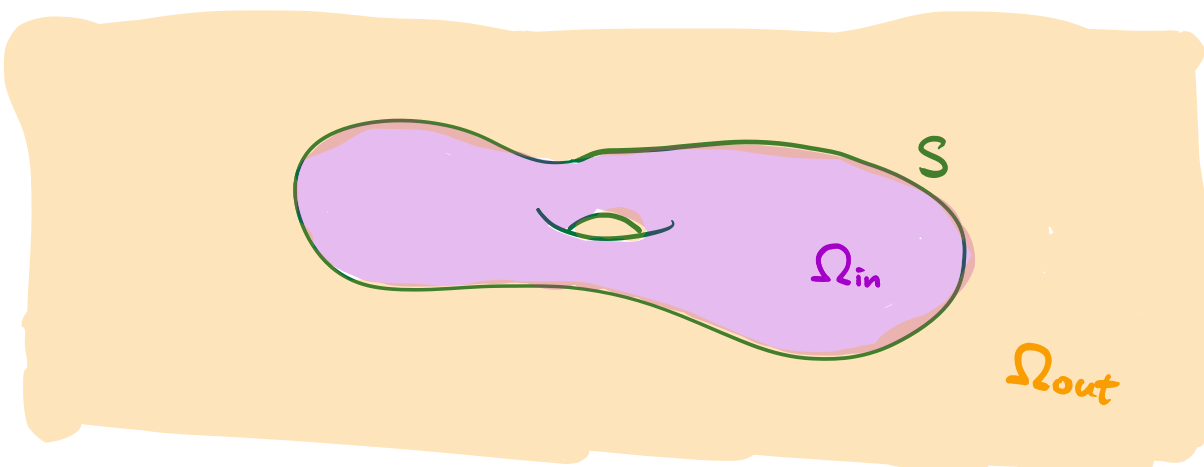
Any connected closed surface $S \subseteq \mathbb{R}^3$ separates $\mathbb{R}^3$ into exactly two connected components:

$$\mathbb{R}^3 \setminus S = \Omega_{in} \cup \Omega_{out}$$

open & connected

s.t. (i) Ω_{in} is bounded while Ω_{out} is unbounded

(ii) $\partial \Omega_{in} = S = \partial \Omega_{out}$



Brouwer-Samelson's Theorem

Any closed surface $S \subseteq \mathbb{R}^3$ is orientable.

$\Rightarrow$ S is homeomorphic to one of the surfaces in $(*)$
(diffeomorphic)